Higher-categorical models of linear logic
PhD defense

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École Polytechnique

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Logic

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$$(A \wedge B) \implies C$$

Proof theory

$$\dfrac{\dfrac{}{A \vdash A}\,(\text{ax}) \qquad \dfrac{}{B \vdash B}\,(\text{ax})}{A, B \vdash A \wedge B}\,(\wedge\text{-R})$$

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Semantics

if A is true and B is true,
then $(A \wedge B)$ is true

Traditional semantics

In traditional semantics: interpret the logic using an ordered set.

Definition
A **model** of traditional logic is:

- an ordered set $(\text{TruthValues}, \leq)$
- for every formula A, a **truth value** $[\![A]\!] \in \text{TruthValues}$
- such that whenever $A \vdash B$, then $[\![A]\!] \leq [\![B]\!]$

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Example
We can take $\text{TruthValues} = \{\text{False}, \text{True}\}$, with $\text{False} < \text{True}$.

Traditional semantics — interpreting formulas

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$[\![A]\!]$	$[\![B]\!]$	$[\![A \land B]\!]$
False	False	False
False	True	False
True	False	False
True	True	True

$[\![A]\!]$	$[\![B]\!]$	$[\![A \implies B]\!]$
False	False	True
False	True	True
True	False	False
True	True	True

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The cut rule

$$\frac{A \vdash B \qquad B \vdash C}{A \vdash C}\,(\text{cut})$$

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is interpreted in the model by the fact that if both

$$[\![A]\!] \leq [\![B]\!] \text{ and } [\![B]\!] \leq [\![C]\!]$$

then

$$[\![A]\!] \leq [\![C]\!]$$

Categorical semantics — interpreting formulas

In categorical semantics, we replace the ordered set TruthValues by a category, for instance sets.

Example

Now $[\![A]\!]$ is no longer True or False, but an arbitrary set.

$$[\![A]\!] = \bullet\bullet\bullet$$

$$[\![B]\!] = \bullet\bullet\bullet\bullet$$

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$$[\![A \vee B]\!] = [\![A]\!] \sqcup [\![B]\!] = \bullet\bullet\bullet \quad \bullet\bullet\bullet\bullet$$

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$$[\![A \implies B]\!] = [\![B]\!]^{[\![A]\!]} = \{f : [\![A]\!] \to [\![B]\!]\}$$

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$$[\![A \implies B]\!] = [\![B]\!]^{[\![A]\!]} = \{f : [\![A]\!] \to [\![B]\!]\}$$

If $[\![A]\!] = \emptyset$, then $[\![A \wedge B]\!] = \emptyset$.

Categorical semantics — interpreting proofs

$$\frac{\overset{p}{\vdots}}{A \vdash B} \qquad \rightsquigarrow \llbracket p \rrbracket : \llbracket A \rrbracket \to \llbracket B \rrbracket$$

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$$\dfrac{\begin{matrix} p \\ \vdots \end{matrix}}{A \vdash B} \qquad \rightsquigarrow [\![p]\!] : [\![A]\!] \to [\![B]\!]$$

$$r \left\{ \dfrac{\dfrac{\begin{matrix} p \\ \vdots \end{matrix}}{A \vdash B} \qquad \dfrac{\begin{matrix} q \\ \vdots \end{matrix}}{B \vdash C}}{A \vdash C} \, (\mathrm{cut}) \right. \qquad \rightsquigarrow [\![r]\!] = [\![q]\!] \circ [\![p]\!]$$

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$$\begin{tikzcd}
\llbracket A \rrbracket \arrow[r, "\llbracket p \rrbracket"] \arrow[rr, bend right, "\llbracket r \rrbracket"'] & \llbracket B \rrbracket \arrow[r, "\llbracket q \rrbracket"] & \llbracket C \rrbracket
\end{tikzcd}$$

Categorical semantics — categories

More generally, we want $[\![A]\!]$ to be any mathematical object for which there is a good notion of “function” or “morphism”.

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Definition
A category $\mathcal{C}$ is the data of:

- Objects (e.g. sets)
- Morphisms (e.g. functions)
- Composition of morphisms
- Such that everything is “well-behaved” (associative composition...)

Categorical semantics — the big picture

Definition
A categorical model in $\mathcal{C}$ is:

- for every formula A, and object $[\![A]\!] \in \mathcal{C}$
- for every proof $p : A \vdash B$, a morphism $[\![p]\!] : [\![A]\!] \to [\![B]\!]$
- compatible with cut
- invariant under **cut elimination**

Traditional vs linear logic

Logic	Traditional	
Formulas	**statements**	
Proof of $A, B \vdash C$	**assuming** A and B, can **prove** C	

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Traditional vs linear logic

Logic	Traditional	Linear
Formulas	**statements**	**ressources**
Proof of $A, B \vdash C$	**assuming** A and B, can **prove** C	**consuming** A and B, can **produce** C

Linear logic — two kinds of “and”

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But $🪙 \nvdash 🥖 \otimes 🥐$.

Linear logic — "unlimited"

A new connective: $!A$ meaning "unlimited A" / "as much A as one wants".

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Example

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$$!(A \mathbin{\&} B) \dashv\vdash\, !A \otimes !B$$

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Example

$$!(A \mathbin{\&} B) \dashv\vdash {!A} \otimes {!B}$$

This leads to two notions of implication:

Linear implication	Traditional implication
$\dfrac{A \vdash B}{\vdash A \multimap B}$	$\dfrac{!A \vdash B}{\vdash A \implies B}$

The relational model of linear logic

Theorem
There is a **model** *of linear logic where:*

- *for every formula* A, $[\![A]\!]$ *is a set*
- *for every proof* $p : A \vdash B$, $[\![p]\!] \subseteq [\![A]\!] \times [\![B]\!]$ *is a* **relation**

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- $[\![A \mathbin{\&} B]\!] = [\![A]\!] \sqcup [\![B]\!]$
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- $[\![!A]\!] = \mathrm{Mul}([\![A]\!]) =$ multisets on $[\![A]\!] =$ lists $(a_1, \ldots, a_n)$ in $[\![A]\!]$ up to reordering

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Can we be more quantitative than relations?

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Can we be more quantitative than relations?

Remark
Given $(a, b) \in [\![A]\!] \times [\![B]\!]$ and $R \subseteq [\![A]\!] \times [\![B]\!]$, either $(a, b) \in R$ or $(a, b) \notin R$.

Quantitative relations

Relation $R \subseteq X \times Y \qquad r : X \times Y \to \{\mathsf{False}, \mathsf{True}\}$

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$$\begin{tikzcd}
& Z \arrow[dl, "f_1"'] \arrow[dr, "f_2"] & \\
X & & Y
\end{tikzcd}$$

A model in spans?

Theorem?
There is a model of linear logic where:

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- for every proof $p : A \vdash B$, $[\![p]\!] : Z \to [\![A]\!] \times [\![B]\!]$ is a span
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WRONG!

$X \mapsto \mathrm{Mul}(X)$ is not **functorial** on spans: it does not preserve the composition of spans.

How to fix this?

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How to fix this?

Lists **up to reordering** are too crude: need to *keep track of symmetries*.

Lists up to reordering

Given $X = \{a, b\}$, lists of size 2 on X up to reordering:

$$(a, a) \qquad (b, b) \qquad (a, b) = (b, a)$$

Lists up to reordering

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If instead of imposing $(a, b) = (b, a)$, we add a "path":

$$\begin{tikzcd}
(a, a) \arrow[loop above, leftrightarrow, "\sim"] & (b, b) \arrow[loop above, leftrightarrow, "\sim"] & (a, b) \arrow[r, leftrightarrow, "\sim"] & (b, a)
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We get a **groupoid** (category with invertible morphisms).

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This idea already underlies the following models:

- Mellies’s span-based template games model.
- Fiore, Gambino, Hyland and Winskel’s generalized species model.

Enter homotopy theory

Sets

$$\mathsf{a} = \mathsf{b}$$

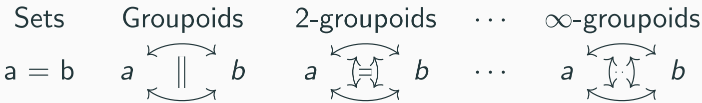
Enter homotopy theory

Sets	Groupoids
a = b	$\begin{tikzcd} a \arrow[r, bend left, leftrightarrow, ""{name=U, below}] \arrow[r, bend right, leftrightarrow, ""{name=D, above}] & b \arrow[from=U, to=D, phantom, "\Vert"] \end{tikzcd}$

Enter homotopy theory

Sets	Groupoids	2-groupoids
$a = b$	$\begin{tikzcd} a \arrow[r, bend left=50, leftrightarrow] \arrow[r, bend right=50, leftrightarrow] \arrow[r, phantom, "\Vert"] & b \end{tikzcd}$	$\begin{tikzcd} a \arrow[r, bend left=50, leftrightarrow] \arrow[r, bend right=50, leftrightarrow] \arrow[r, phantom, "\Updownarrow = \Updownarrow"] & b \end{tikzcd}$

Enter homotopy theory



Sets Groupoids 2-groupoids ⋯ ∞-groupoids

$a = b$

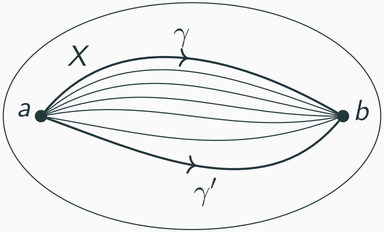
Enter homotopy theory

Sets Groupoids 2-groupoids ··· ∞-groupoids

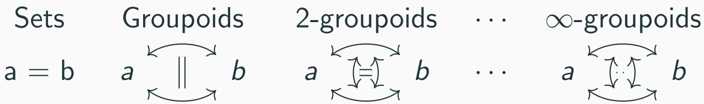
$a = b$

Remark

In topology: spaces have points, paths, deformations of paths...



Enter homotopy theory



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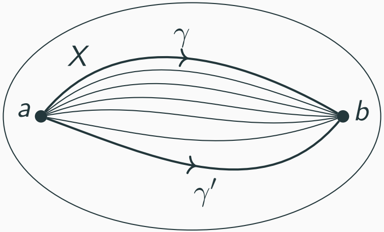

∞-groupoids ≈ spaces up to *homotopy*

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Homotopy type theory

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- an alternative foundation to set theory
- based on Martin-Löf's type theory
- a formal language to speak about ∞-groupoids

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- a formal language to speak about ∞-groupoids

Set theory	set X	elements $a, b \in X$	$a = b$ is either True or False
Type theory	type X	elements $a, b : X$	$a = b$ is itself a type

- $a = b$ can have multiple elements.
- given $p, q : a = b$, can form the type $p = q$, and so on.

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$\rightsquigarrow$ types have ∞-groupoid structure.

Homotopy multisets

Goal: “homotopify” multisets.

$$\mathrm{Mul}(X) = \bigsqcup_{n \in \mathbb{N}} X^n / \mathfrak{S}_n$$

Homotopy multisets

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$$\mathsf{Mul}(X) = \bigsqcup_{n\in\mathbb{N}} X^n/\mathfrak{S}_n$$

Definition
In HoTT, the type of **homotopy multisets** on a type X is

$$\mathsf{HMul}(X) = \sum_{E:\mathsf{FinSet}} X^E$$

Elements of HMul(X): pairs (E, f) where:

- E finite set
- $f : E \to X$

Homotopy multisets examples

Equalities $(E, p) = (F, q)$ are given by

$$\begin{tikzcd}
E \arrow[rr, "f", "\sim"'] \arrow[dr, "p"'] & & F \arrow[dl, "q"] \\
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If $X = \{\textcolor{blue}{\bullet}, \textcolor{red}{\bullet}\}$ and $E = \{0, 1\}$,

$$\{\textcolor{blue}{0}, \textcolor{blue}{1}\} \qquad \{\textcolor{red}{0}, \textcolor{red}{1}\} \qquad \{\textcolor{red}{0}, \textcolor{blue}{1}\} \qquad \{\textcolor{blue}{0}, \textcolor{red}{1}\}$$

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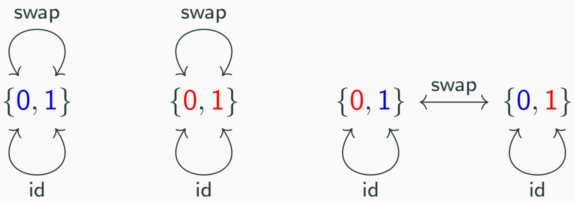

A span-based model of linear logic in HoTT

Theorem (H, Mimram 2024)
In homotopy type theory, there is a **Seely category** Span *with:*

- *objects are types*
- *morphisms are spans* $X \leftarrow Z \rightarrow Y$
- $[\![A \,\&\, B]\!] := [\![A]\!] \sqcup [\![B]\!]$
- $[\![A \otimes B]\!] := [\![A]\!] \times [\![B]\!]$
- $[\![!A]\!] := \mathsf{HMul}([\![A]\!])$

Linear logic and Seely categories

Definition (Seely category)

1. symmetric monoidal category $(\mathcal{C}, \otimes, 1, \multimap)$
2. with finite products ($\&$ and $\top$),
3. a comonad $(!, \delta, \varepsilon) : \mathcal{C} \to \mathcal{C}$,
4. isomorphisms $m^2_{X,Y} : !(X \mathbin{\&} Y) \simeq !X \otimes !Y$ (recall $!(A \mathbin{\&} B) \dashv\vdash !A \otimes !B$)
   $m^0 : !\top \simeq 1$
5. commutative diagram:

$$\begin{tikzcd}
!X \otimes !Y \arrow[rr, "\delta_X \otimes \delta_Y"] \arrow[d, "m^2_{X,Y}"'] & & !!X \otimes !!Y \arrow[d, "m^2_{!X,!Y}"] \\
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Theorem (Seely)
Every Seely category is a **model of linear logic**.

Kleisli category of a Seely category

Proposition
From a Seely category $\mathcal{C}$, can build its **Kleisli category** $\mathcal{C}_!$ with:

- the same objects
- morphisms $X \to Y$ in $\mathcal{C}_!$ are morphisms $!X \to Y$ in $\mathcal{C}$

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Theorem
$\mathcal{C}_!$ is a model of **traditional logic***.*

Linear implication *Traditional implication*

$$\frac{A \vdash B}{\vdash A \multimap B} \qquad \frac{!A \vdash B}{\vdash A \implies B}$$

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What does $\mathsf{Span}_{\mathsf{HMul}}$ look like?

Non-linear spans are polynomials

A polynomial (in types) is a diagram

$$\begin{tikzcd}
& E \arrow[r, "p"] \arrow[dl, "s"'] & B \arrow[dr, "t"] & \\
I & & & J
\end{tikzcd}$$

Theorem (H, Mimram 2024)
Poly_{fin} *is the Kleisli category for the comonad* HMul *on* Span*:*

$$\mathrm{Span}(\mathrm{HMul}(I), J) \simeq \mathrm{Poly}_{fin}(I, J)$$

where $\mathrm{HMul}(X) = \sum_{E:\mathrm{FinSet}}(E \to X)$*.*

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Remark: polynomial functors

Any polynomial

$$I \xleftarrow{s} E \xrightarrow{p} B \xrightarrow{t} J$$

induces a **polynomial functor**

$$\begin{aligned} \mathcal{U}^I &\to \mathcal{U}^J \\ (X_i)_{i\in I} &\mapsto \Big(\sum_{B\in t^{-1}(j)} \prod_{e\in p^{-1}(b)} X_{s(e)} \Big)_{j\in J} \end{aligned}$$

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Remark

When the polynomial is a span (i.e. p is an isomorphism),

$$(X_i)_{i \in I} \mapsto \Big(\sum_{B \in t^{-1}(j)} X_{s(p^{-1}(e))} \Big)_{j \in J}$$

A parenthesis on differential linear logic

Differential linear logic extends linear logic based on the following analogy:

Differential calculus	Linear logic
"Every linear map is smooth"	$\dfrac{A \vdash B}{!A \vdash B}$ (der)
Every smooth $f : \mathbb{R}^m \to \mathbb{R}^n$ has a differential $df_0 : \mathbb{R}^m \to \mathbb{R}^n$	

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Theorem (H, unpublished)
The Seely category $(\mathrm{Span}, \mathrm{HMul}_{\mathcal{V}})$ *is a model of differential linear logic whenever the types in* $\mathcal{V}$ *are* **discrete** *(e.g.* $\mathcal{V} = \mathrm{FinSet}$ *works).*

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Insight from homotopy theory: higher coherences

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In an ∞-**category**, composition is **homotopy coherently** associative.

Recall: Seely categories

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Point 5 is too ad hoc ⇝ **no natural ∞-categorical generalization**.

Linear/non-linear adjunctions

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A *linear/non-linear adjunction* is an adjunction

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All readily make sense for ∞-categories!

∞-linear/non-linear adjunction

Theorem (Benton)
A categorical model of linear logic is an LNL adjunction between categories.

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∞-linear/non-linear adjunction

Definition
An ∞-categorical model of linear logic is an LNL adjunction between ∞-categories.

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Sanity check: Seely isomorphisms

We should make sure we still have the Seely isomorphisms $!(A \mathbin{\&} B) \dashv\vdash {!A} \otimes {!B}$.

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Theorem (H, Mimram 2025)

In a linear/non-linear adjunction $(\mathcal{M}, \times) \underset{M}{\overset{L}{\rightleftarrows}} (\mathcal{L}, \otimes)$, *we have*

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Proof.
Right adjoints preserve products, so $M(X \mathbin{\&} Y) \simeq M(X) \mathbin{\&} M(Y)$.

Since, $L : (\mathcal{M}, \&) \to (\mathcal{L}, \otimes)$ is strongly monoidal, we have

$$LM(X \mathbin{\&} Y) \simeq L(M(X) \mathbin{\&} M(Y)) \simeq LM(X) \otimes LM(Y). \qquad \square$$

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Proposition

In a model of linear logic, every $!X$ has a canonical **commutative comonoid** structure.

Proof.

Comes from $!A \vdash\, !A \otimes\, !A$, and cut-elimination invariance. ☐

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Theorem (H, Mimram 2025)
In an LNL adjunction, every $LM(X)$ *has a canonical commutative comonoid structure.*

Proof.
In an ∞-category with finite products, every object admits a **unique** commutative comonoid structure $\Delta : X \to X \times X$.

$\mathcal{M}$ has finite products, so every $M(X)$ is a commutative comonoid in $\mathcal{M}$.

$L : \mathcal{M} \to \mathcal{L}$ is strongly monoidal, so it preserves commutative comonoids.

☐

A special case : Lafont (∞-)categories

Theorem (Lafont)
If for every $X \in \mathcal{L}$, there exists a **universal** *commutative comonoid $!_u X$ in $\mathcal{L}$, then*

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defines a model of linear logic.

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What about for ∞-categories?

Theorem (H, Mimram 2025)
If $\mathcal{L}$ admits universal commutative comonoids, then the forgetful functor $\mathrm{Comon}(\mathcal{L}) \to \mathcal{L}$ *has a right adjoint, and this forms a linear/non-linear adjunction*

$$\begin{tikzcd}(\mathrm{Comon}(\mathcal{L}), \times) \arrow[r, shift left] \arrow[r, phantom, "\perp"] & (\mathcal{L}, \otimes) \arrow[l, shift left]\end{tikzcd}.$$

An explicit formula for universal comonoids

The following has been shown in 1-category theory by Mellies, Tabareau, Tasson.

Theorem (H, Mimram 2025)
Let $(\mathcal{L}, \otimes)$ be a symmetric monoidal ∞-category, and $X \in \mathcal{L}$. If for all $A \in \mathcal{L}$,

$$A \otimes \prod_{n \in \mathbb{N}} (X^{\otimes n})^{\mathfrak{S}_n} \to \prod_{n \in \mathbb{N}} (A \otimes X^{\otimes n})^{\mathfrak{S}_n}$$

is an isomorphism, then

$$\prod_{n \in \mathbb{N}} (X^{\otimes n})^{\mathfrak{S}_n}$$

is the universal commutative comonoid on X.

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Proof.
It follows from more general dual results of Lurie on free algebras for ∞-operads. ☐

Another criterion for existence of universal comonoids

An ∞-category $\mathcal{C}$ is *presentable* if:

- it admits small colimits
- it is *generated* under *filtered colimits* by a *small set* of *compact* objects

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Theorem (H, Mimram 2025)
Let $\mathcal{C}$ be a symmetric monoidal presentable ∞-category such that $\forall X \in \mathcal{C}$, the functor

$$X \otimes - : \mathcal{C} \to \mathcal{C}$$

preserves small colimits. Then $\mathcal{C}$ admits universal commutative comonoids.

But in general there is no nice formula in this context.

Example: ∞-categorical generalized species

Theorem (H, Mimram 2025)
The following ∞-category admits universal commutative comonoids:

- *the objects are ∞-categories $\mathcal{C}, \mathcal{D}, \ldots$*
- *the morphisms are ∞-profunctors $\mathcal{C} \times \mathcal{D}^{op} \to \infty\mathrm{Grpd}$*

And $!_u\mathcal{C}$ is given by the free symmetric monoidal ∞-category on $\mathcal{C}$.

Proof.
The proof relies on the criterion for the explicit formula $\prod_{n\in\mathbb{N}}(X^{\otimes n})^{\mathfrak{S}_n}$. □

Comparison of higher relations

	Relations	Spans (HoTT)	Profunctors
$[\![A]\!]$	Sets X, Y	∞-groupoids X, Y	∞-categories $\mathcal{C}, \mathcal{D}$
$[\![A \multimap B]\!]$	$R \subseteq X \times Y$ $X \times Y \to \{\text{False}, \text{True}\}$	$Z \to X \times Y$ $X \times Y \to \infty\text{Grpd}$	two-sided discrete fibrations $\mathcal{C} \times \mathcal{D}^{\text{op}} \to \infty\text{Grpd}$
$[\![A \implies B]\!]$		Polynomials	Generalized species

Example: abelian “things” — 1

Proposition
There is a linear/non-linear adjunction of 1-categories

$$\begin{tikzcd}(\mathrm{Set}, \times) \arrow[r, shift left, "\mathbb{Z}\langle - \rangle"] \arrow[r, phantom, "\perp"] & (\mathrm{Ab}, \otimes) \arrow[l, shift left]\end{tikzcd}$$

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Theorem (H, Mimram 2025)
There is a linear/non-linear adjunction of ∞-categories

$$\begin{tikzcd}(\infty\mathsf{Grpd}, \times) \arrow[r, shift left, "\mathbb{S}\langle - \rangle"] \arrow[r, phantom, "\perp"] & (\mathsf{Sp}, \otimes) \arrow[l, shift left]\end{tikzcd}$$

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Example: abelian “things” — 2

Proposition
Ab admits universal commutative comonoids.

Example: abelian “things” — 2

Proposition
Ab admits universal commutative comonoids.

Theorem (H, Mimram 2025)
Sp *admits universal commutative comonoids.*

Remark
More generally, this works for vector spaces, modules, module spectra...

Summary

- We generalized categorical semantics of linear logic to the homotopical/higher setting
- Gave a family of interprations for ! in categories of spans in HoTT
- We constructed several ∞-categorical models generalizing well-known 1- and 2-categorical ones (relations, species, vector spaces, abelian groups)

Future work

- Give direct definitions of linear ∞-categories and Seely ∞-categories, and show they induce LNL adjunctions.
- Compare the HoTT approach with the ∞-categorical one.
- Generalize Mellies' span model (template games) to this new setting (in connection with polynomial functors).
- Generalize to $(\infty, 2)$-categorical setting to model differential linear logic.
- Try to fit advanced homotopical constructions with linear flavour (Goodwillie calculus?) into this new setting.